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ACOUSTIC DESTABILIZATION OF VORTICES

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CONTENTS

A line vortex which has uniform vorticity $2\Omega_0$ in its core is subjected to a small two-dimensional disturbance whose dependence on polar angle is $e^{im\theta}$. The stability is examined according to the equations of compressible, inviscid flow in a homentropic medium. The boundary condition at infinity is that of outgoing acoustic waves, and it is found that this capacity to radiate leads to a slow instability by comparison with the corresponding incompressible vortex which is stable. Numerical eigenvalues are computed as functions of the mode number m and the Mach number M based on the circumferential speed of the vortex. These are compared with an asymptotic analysis for the $m = 2$ mode at low Mach number in which it is found that the growth rate is $(\pi/32)\,M^4\Omega_0$ in good agreement with the numerical results.

1. Introduction

When a straight vortex filament, with uniform vorticity in its core is slightly disturbed, it will oscillate. This was shown by Lord Kelvin (1880). In a compressible fluid these oscillations can excite sound waves which will carry energy away to infinity. If the amplitude of the oscillations is prescribed and the vortex filament regarded as a compact sound source, the intensity of the acoustic radiation can be calculated from the Lighthill theory (Broadbent 1976). The period of oscillation of the vortex filament is of the order of the rotation period of fluid particles in the vortex core, so that the compactness condition, necessary for the Lighthill theory to be applicable, is satisfied if the Mach number based on the maximum velocity of swirling is small. However, this condition will not be satisfied for the vortex filaments which form in a supersonic mixing layer, so that it is of interest to consider the radiation without imposing the compactness condition.

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In the Lighthill theory, the flow producing the noise is regarded as being prescribed and modifications to this flow due to the radiation are ignored. For the special class of flows considered here, namely infinitesimal disturbances to a vortex filament in a compressible inviscid fluid, we can avoid both these assumptions and our results are exact. However, we have not considered nonlinear effects. We neglect viscosity and thermal conductivity and consider the undisturbed vortex to be specified by the swirl velocity $\bar{V}(r)$ and the entropy $\bar{S}(r)$ where r is distance from the axis of the vortex. Once these distributions are fixed the pressure, density and local sound speed can be determined. The equations governing small disturbances to this vortex are derived in § 2 and are shown to lead to an energy-like integral constraint. We consider only one Fourier component of the disturbance, which is thus characterized by an azimuthal wave number m, where m takes positive integral values and a corresponding angular frequency ω. The angular frequency will be determined by demanding that the disturbance satisfies the equations of motion and the radiation condition.

This eigenvalue problem is set up in § 3, for the special case in which the entropy $\bar{S}$ is uniform and there is uniform vorticity in the vortex core. The great advantage of this choice of swirl velocity distribution is that the results of our analysis reduce to Kelvin's in the incompressible limit and, in particular, we know that in this limit the vortex is stable. We cannot, without detailed calculation, tell whether a vortex will be unstable to *non*-axisymmetric disturbances even if the Rayleigh criterion shows it to be stable to axisymmetric disturbances. We have picked uniform entropy for want of any information about the entropy distribution in actual flows; it is clear, however, from the form of the equations that entropy gradients can be significant. We summarize Kelvin's results in § 4 and describe our numerical procedure and results in § 5. We define a Mach number M based on the swirl velocity and local sound speed at the vortex core boundary and we present our computed values of $\omega(m, M)$. The most striking feature of these results is that for all $M > 0$ and all the values of m considered $\operatorname{Im}(\omega) < 0$, corresponding to instability of the flow. This instability has been introduced by the ability of the system to radiate, which in turn destroys the $\pi/2$ out-of-phase relation in the incompressible Reynolds stress. The effect is small, the e-folding time of the instability being at least 15 rotation periods. It is surprising that the radiative energy loss does not cause the oscillations to decay. The growing oscillations must draw energy from the mean flow; however, a proper discussion of the energetics requires solution of the governing equation to second order in the disturbance amplitude and we have not attempted this.

When the Mach number is small, the eigenfrequency can be found by a perturbation calculation. The perturbation about Kelvin's solution is singular, because the disturbance at large distances merges into an acoustic wave however small M is. The analysis is given in § 6 and shows that for $m = 2$

$$\operatorname{Im}(\omega) = -\tfrac{1}{32}\pi\,\Omega_0 M^4 + O(M^6 \ln M),$$

where $2\pi/\Omega_0$ is the rotation period of the core.

The stability of a compressible swirling flow confined between coaxial circular cylinders was examined by Howard & Gupta (1962). They were interested in the extension of the Rayleigh criterion and thus they considered axi-symmetric three-dimensional disturbances. The confined nature of the flow prevents acoustic radiation which may explain why Howard and Gupta did not find an instability of the type we have discovered. However, we have not found any eigen-solutions when $m = 0$, so we do not know if there would be instability in this case.

2. The disturbance equations

We use plane polar coordinates (r, θ). Then if the velocity is $v_r\,\hat{\boldsymbol{r}}+v_\theta\,\hat{\boldsymbol{\theta}}$, the equations of motion are

$$\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{2.1}$$

$$\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}=-\frac{1}{\rho r}\frac{\partial p}{\partial \theta}, \tag{2.2}$$

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)=0, \tag{2.3}$$

and

$$\frac{\partial S}{\partial t}+v_r\frac{\partial S}{\partial r}+\frac{v_\theta}{r}\frac{\partial S}{\partial \theta}=0, \tag{2.4}$$

where ρ is the density, p is the pressure and S is the specific entropy; for a calorically perfect gas with constant specific heats c_{v} and c_{p}

$$S=c_{\mathrm{v}}\ln p-c_{\mathrm{p}}\ln\rho+S_0, \tag{2.5}$$

where S_0 is a constant. As usual, we write $\gamma=c_{\mathrm{p}}/c_{\mathrm{v}}$.

The unperturbed vortex is defined by

$$\left.\begin{aligned} v_\theta &= \bar{V}(r),\\ v_r &= 0,\end{aligned}\right\} \tag{2.6}$$

and

$$S=\bar{S}(r). \tag{2.7}$$

These expressions satisfy the full equations of motion provided that

$$\rho=\bar{\rho}(r) \tag{2.8}$$

and

$$p=\bar{p}(r), \tag{2.9}$$

where

$$\frac{\bar{V}^2}{r}=\frac{1}{\bar{\rho}}\frac{\mathrm{d}\bar{p}}{\mathrm{d}r} \tag{2.10}$$

and

$$\bar{S}=c_{\mathrm{v}}\ln\bar{p}-c_{\mathrm{p}}\ln\bar{\rho}+S_0. \tag{2.11}$$

Clearly once $\bar{V}(r)$ and $\bar{S}(r)$ are given, $\bar{p}(r)$ and $\bar{\rho}(r)$ are determined.

We suppose that the vortex is slightly perturbed and write

$$\left.\begin{aligned} v_r &= \tilde{v}_r(r,\theta,t),\\ v_\theta &= \bar{V}(r)+\tilde{v}_\theta(r,\theta,t),\\ p &= \bar{p}(r)+\tilde{p}(r,\theta,t),\\ \rho &= \bar{\rho}(r)+\tilde{\rho}(r,\theta,t),\\ \text{and}\quad S &= \bar{S}(r)+\tilde{s}(r,\theta,t).\end{aligned}\right\} \tag{2.12}$$

If we substitute (2.12) into the governing equations (2.1)–(2.5) we get after linearizing

$$\frac{\partial \tilde{v}_r}{\partial t}+\frac{\bar{V}}{r}\frac{\partial \tilde{v}_r}{\partial \theta}-\frac{2\bar{V}}{r}\tilde{v}_\theta=\frac{\mathrm{d}\bar{p}}{\mathrm{d}r}\frac{\tilde{\rho}}{\bar{\rho}^2}-\frac{1}{\bar{\rho}}\frac{\partial \tilde{p}}{\partial r}, \tag{2.13}$$

$$\frac{\partial \tilde{v}_\theta}{\partial t}+\frac{\bar{V}}{r}\frac{\partial \tilde{v}_\theta}{\partial \theta}+\left(\frac{\mathrm{d}\bar{V}}{\mathrm{d}r}+\frac{\bar{V}}{r}\right)\tilde{v}_r=-\frac{1}{\bar{\rho}r}\frac{\partial \tilde{p}}{\partial \theta}, \tag{2.14}$$

$$\frac{\partial \tilde{\rho}}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\bar{\rho}\tilde{v}_r)+\frac{\bar{\rho}}{r}\frac{\partial \tilde{v}_\theta}{\partial \theta}+\frac{\bar{V}}{r}\frac{\partial \tilde{\rho}}{\partial \theta}=0, \tag{2.15}$$

$$\frac{\partial \tilde{s}}{\partial t}+\frac{\bar{V}}{r}\frac{\partial \tilde{s}}{\partial \theta}+\frac{\mathrm{d}\bar{S}}{\mathrm{d}r}\tilde{v}_r=0, \tag{2.16}$$

and

$$\tilde{s}=\frac{c_\mathrm{v}}{\bar{p}}(\tilde{p}-a_0^2\tilde{\rho}), \tag{2.17}$$

where $a_0^2(r)=\gamma\bar{p}(r)/\bar{\rho}(r)$, so that $a_0(r)$ is the local sound speed in the unperturbed vortex.

Before going on to discuss how the linearized equations are to be solved, we note that the solution of these equations satisfies an energy-like integral constraint. To deduce this, we multiply equation (2.13) by $\bar{\rho}\tilde{v}_r$ and multiply equation (2.14) by $\bar{\rho}\tilde{v}_\theta$ and add. After some algebra we find that

$$\mathrm{D}(\tfrac{1}{2}\bar{\rho}\tilde{v}_r^2+\tfrac{1}{2}\bar{\rho}\tilde{v}_\theta^2)+\bar{\rho}\tilde{v}_r\tilde{v}_\theta\left(\frac{\mathrm{d}\bar{V}}{\mathrm{d}r}-\frac{\bar{V}}{r}\right)=-\frac{1}{r}\frac{\partial}{\partial r}(r\tilde{v}_r\tilde{p})-\frac{1}{r}\frac{\partial}{\partial \theta}(\tilde{v}_\theta\tilde{p})+\frac{\tilde{p}\tilde{v}_r}{\bar{\rho}}\frac{\mathrm{d}\bar{p}}{\mathrm{d}r}+\tilde{p}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tilde{v}_r)+\frac{1}{r}\frac{\partial \tilde{v}_\theta}{\partial \theta}\right), \tag{2.18}$$

where

$$\mathrm{D}\equiv\frac{\partial}{\partial t}+\frac{\bar{V}}{r}\frac{\partial}{\partial \theta}.$$

However, the linearized equation of continuity (2.15) shows that

$$\frac{1}{r}\frac{\partial}{\partial r}(r\tilde{v}_r)+\frac{1}{r}\frac{\partial \tilde{v}_\theta}{\partial \theta}=-\frac{1}{\bar{\rho}}\mathrm{D}\tilde{\rho}-\frac{\tilde{v}_r}{\bar{\rho}}\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}r}. \tag{2.19}$$

If this is substituted into (2.18) and equation (2.17) used to eliminate $\tilde{\rho}$ in favour of $\tilde{p}$ and $\tilde{s}$, equation (2.18) becomes

$$\mathrm{D}\tilde{E}=-\bar{\rho}\tilde{v}_r\tilde{v}_\theta\left(\frac{\mathrm{d}\bar{V}}{\mathrm{d}r}-\frac{\bar{V}}{r}\right)-\frac{\tilde{v}_r\tilde{s}}{c_\mathrm{p}}\frac{\mathrm{d}\bar{p}}{\mathrm{d}r}-\frac{1}{r}\frac{\partial}{\partial r}(r\tilde{v}_r\tilde{p})-\frac{1}{r}\frac{\partial}{\partial \theta}(\tilde{v}_\theta\tilde{p}), \tag{2.20}$$

where we have defined

$$\tilde{E}=\tfrac{1}{2}\bar{\rho}(\tilde{v}_r^2+\tilde{v}_\theta^2)+\frac{1}{2}\frac{\tilde{p}^2}{\bar{\rho}a_0^2}. \tag{2.21}$$

If we define an angular average by

$$\langle\psi\rangle=\frac{1}{2\pi}\int_0^{2\pi}\psi(\theta)\,\mathrm{d}\theta, \tag{2.22}$$

then application of this average to (2.20) reduces it to the form

$$\frac{\partial\langle\tilde{E}\rangle}{\partial t}=\bar{\rho}\langle-\tilde{v}_r\tilde{v}_\theta\rangle\left(\frac{\mathrm{d}\bar{V}}{\mathrm{d}r}-\frac{\bar{V}}{r}\right)+\frac{\langle-\tilde{v}_r\tilde{s}\rangle}{c_\mathrm{p}}\frac{\mathrm{d}\bar{p}}{\mathrm{d}r}-\frac{1}{r}\frac{\partial}{\partial r}(r\langle\tilde{v}_r\tilde{p}\rangle), \tag{2.23}$$

or, after integration

$$\frac{\partial}{\partial t}\int_0^{r_1}\langle\tilde{E}\rangle r\,\mathrm{d}r=\int_0^{r_1}\bar{\rho}\langle-\tilde{v}_r\tilde{v}_\theta\rangle\left(\frac{\mathrm{d}\bar{V}}{\mathrm{d}r}-\frac{\bar{V}}{r}\right)r\,\mathrm{d}r+\int_0^{r_1}\frac{\langle-\tilde{v}_r\tilde{s}\rangle}{c_\mathrm{p}}\frac{\mathrm{d}\bar{p}}{\mathrm{d}r}r\,\mathrm{d}r-[r\langle\tilde{v}_r\tilde{p}\rangle]_{r=r_1}. \tag{2.24}$$

It must be emphasized that this integral is a deduction from the linearized equations and, while $\hat{E}$ is what is described as the acoustic energy for sound waves propagating in a uniform gas at rest, we are not claiming that (2.23) represents an energy balance for the disturbed vortex. The deduction of an energy balance would require consideration of the governing equations retaining terms of second order in amplitude of the disturbance. However, $\langle\tilde{E}\rangle$ is a positive definite functional of the disturbance and is thus a convenient measure of its amplitude. We shall use (2.24) later, both as a check on the numerical solution and to provide an indication of the source of the instability; for homentropic flow ($\mathrm{d}\bar{S}/\mathrm{d}r = 0$) equation (2.6) implies that $\tilde{s} = 0$, so that the entropy contribution vanishes in equations (2.20)–(2.24).

3. THE EIGENVALUE PROBLEM

We are interested in calculating the eigenfrequencies for the vortex. Thus we write

$$\tilde{v}_r(r, \theta, t) = \tilde{v}_r(r) \exp\left[\mathrm{i}(\omega t + m\theta)\right], \tag{3.1}$$

with a similar assumption for the other disturbance quantities; use of the same symbol for a disturbance quantity and its Fourier coefficient will not lead to confusion, because we will consider only the particular θ and t dependence embodied in (3.1) from this point on. We restrict m to be a positive integer or zero. This involves no loss of generality and simplifies the presentation of the results. If the substitution (3.1) is made throughout the linearized governing equations (2.13)–(2.17) a system of ordinary differential equations results and this system can be condensed into a pair of equations for $\tilde{p}(r)$ and $\tilde{v}_r(r)$. This pair of equations is

$$\frac{\partial \tilde{p}}{\partial r} = \tilde{p}\left[\frac{\bar{V}^2}{r a_0^2} - \frac{2m\bar{V}}{r^2\sigma}\right] + \bar{p}\tilde{v}_r\left[\frac{\bar{V}^2}{\mathrm{i}\sigma c_\mathrm{p} r}\frac{\mathrm{d}\bar{S}}{\mathrm{d}r} - \mathrm{i}\sigma - \frac{2\bar{V}\varpi}{\mathrm{i}\sigma r}\right], \tag{3.2}$$

and

$$\frac{\partial \tilde{v}_r}{\partial r} = \frac{\mathrm{i}\sigma\tilde{p}}{\bar{p}}\left[\frac{m^2}{\sigma^2 r^2} - \frac{1}{a_0^2}\right] + \tilde{v}_r\left[\frac{m\varpi}{\sigma r} - \frac{\bar{V}^2}{r a_0^2} - \frac{1}{r}\right], \tag{3.3}$$

where

$$\sigma = \omega + \frac{m\bar{V}(r)}{r}, \tag{3.4}$$

and

$$\varpi = \frac{\mathrm{d}\bar{V}}{\mathrm{d}r} + \frac{\bar{V}}{r} \tag{3.5}$$

is the vorticity in the undisturbed vortex.

This system has a regular singular point at $r = 0$ introduced by the polar coordinates and singular points where $\sigma(r_\mathrm{c}) = 0$ or

$$\omega = -m\bar{V}(r_\mathrm{c})/r_\mathrm{c}.$$

This corresponds to a disturbance propagating around the circle $r = r_\mathrm{c}$ at just the swirl velocity. This means that there is a critical layer in the solution; however, the singular point lies in the irrotational part of the flow, and is at a value of r with a small imaginary part since ω is complex. The solution is thus analytic at $r = r_\mathrm{c}$, and the singularity from the governing equations can be removed by a change of variable; in particular there is no need to invoke viscosity to determine the phase of the inviscid solution.†

All the detailed analysis has been done for a vortex with uniform entropy and with vorticity that is uniform in $0 < r < a$ and zero for $r > a$. Hence

$$\bar{V}(r) = \begin{cases} \Omega r & r < a, \\ \Omega a^2/r & r > a. \end{cases} \tag{3.6}$$

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† We are grateful to Professor J. T. Stuart, F.R.S. for a discussion of this point.

We change the equations to non-dimensionalized form using the values of the physical variables at $r = a$. Thus

$$\left.\begin{aligned} r &= ax, \\ \bar{V}(r) &= \Omega_0 a \hat{V}(x), \\ a_0(r) &= a_0(a)\,\hat{c}(x), \\ \sigma(r) &= \Omega_0\,\hat{\sigma}(x), \\ \omega &= \Omega_0 f, \\ \bar{\omega}(r) &= \Omega_0\,\hat{\omega}(x), \\ \text{and} \qquad \bar{\rho}(r) &= \bar{\rho}(a)\,\hat{\rho}(x), \end{aligned}\right\} \tag{3.7}$$

while new dependent variables are defined by

$$\left.\begin{aligned} \tilde{v}_r(r) &= \Omega_0 a U, \\ \text{and} \qquad \tilde{p}(r) &= \bar{p}(a)\,\Omega_0^2 a^2 P. \end{aligned}\right\} \tag{3.8}$$

This leads to the equations

$$\left.\begin{aligned} \mathrm{d}P/\mathrm{d}x &= A(x)\,P + B(x)\,U, \\ \text{and} \qquad \mathrm{d}U/\mathrm{d}x &= C(x)\,P + D(x)\,U, \end{aligned}\right\} \tag{3.9}$$

where
$$A(x) = \frac{\hat{V}^2 M^2}{x\hat{c}^2} - \frac{2m\hat{V}}{x^2\hat{\sigma}}, \tag{3.10}$$

$$B(x) = \mathrm{i}\hat{\rho}\left(-\hat{\sigma} + \frac{2\hat{V}\hat{\omega}}{x\hat{\sigma}}\right), \tag{3.11}$$

$$C(x) = \frac{\mathrm{i}}{\hat{\rho}}\left(\frac{m^2}{x^2\hat{\sigma}} - \frac{M^2\hat{\sigma}}{\hat{c}^2}\right), \tag{3.12}$$

and
$$D(x) = \frac{m\hat{\omega}}{\hat{\sigma}x} - \frac{\hat{V}^2 M^2}{x\hat{c}^2} - \frac{1}{x}. \tag{3.13}$$

Here M is the Mach number characterizing the motion of the boundary of the vortex, so that

$$M = \Omega a / a_0(a). \tag{3.14}$$

The quantity $\hat{\sigma}$ is given by
$$\hat{\sigma} = f + m\hat{V}/x, \tag{3.15}$$

and integration of the pressure balance equation (2.10) shows that the dimensionless local sound speed $\hat{c}$ is given by

$$\hat{c}^2(x) = \begin{cases} 1 - \frac{1}{2}(\gamma - 1)\,M^2(1 - x^2) & (x < 1), \\ 1 + \frac{1}{2}(\gamma - 1)\,M^2(1 - x^{-2}) & (x > 1), \end{cases} \tag{3.16}$$

and the local density $\hat{\rho}$ by
$$\hat{\rho}(x) = [\hat{c}^2(x)]^{1/(\gamma - 1)}. \tag{3.17}$$

Clearly $\hat{c}$ and $\hat{\rho}$ are positive for $x > 1$, whatever M, but $\hat{c}$ and $\hat{\rho}$ can vanish for $x < 1$. This restricts the Mach number and

$$M < \left(\frac{2}{\gamma - 1}\right)^{\frac{1}{2}} = 2.2360\ldots, \tag{3.18}$$

when $\gamma = 1.4$.

The eigenfrequency f will be determined by solving the system (3.9) subject to the requirements that P and U are well-behaved at $x = 0$ and that P and U correspond to outward propagating waves at $x = \infty$.

The origin can be shown to be a regular singular point of the system (3.9) at which the well-behaved solutions are characterized by

$$\left.\begin{aligned} P &\sim x^m, \\ U &\sim x^{m-1}, \end{aligned} \quad \text{as} \quad x \to 0.\right\} \tag{3.19}$$

This leads to expansions

$$\left.\begin{aligned} P &= P_0 x^m + P_2 x^{m+2} + \ldots, \\ \text{and} \qquad U &= U_0 x^{m-1} + U_2 x^{m+1} + \ldots, \end{aligned}\right\} \tag{3.20}$$

and substitution into the system (3.9) enables all the coefficients to be determined in terms of P_0. These starting series are used in the numerical integrations described in § 4.

For large x,

$$\left.\begin{aligned} A &= O\left(\frac{1}{x^3}\right), \\ B &= -\mathrm{i}f\hat{\rho}(\infty)\left[1+\left(\frac{m}{f}-\frac{\beta}{\gamma-1}\right)\frac{1}{x^2}\right]+O\left(\frac{1}{x^4}\right), \\ C &= -\frac{\mathrm{i}M^2 f}{\hat{c}(\infty)^2\hat{\rho}(\infty)}\left[1+\left(\frac{m}{f}+\frac{\gamma\beta}{\gamma-1}-\frac{m^2\hat{c}(\infty)^2}{M^2f^2}\right)\frac{1}{x^2}\right]+O\left(\frac{1}{x^4}\right), \\ \text{and} \qquad D &= -\frac{1}{x}+O\left(\frac{1}{x^3}\right), \end{aligned}\right\} \tag{3.21}$$

where

$$\beta = \frac{\frac{1}{2}(\gamma-1)\,M^2}{1+\frac{1}{2}(\gamma-1)\,M^2}. \tag{3.22}$$

If terms of order $1/x^2$ are ignored the governing equations (3.9) are identical with those for a uniform medium at rest. This suggests that P and U will have expansions of the form

$$\begin{aligned} P &= \left[p^* x^{-\frac{1}{2}} \exp\left(-\frac{\mathrm{i}fxM}{\hat{c}(\infty)}\right)\right]\left(1+\frac{p_1}{x}+\frac{p_2}{x^2}+\ldots\right), \\ \text{and} \qquad U &= \left[u^* x^{-\frac{1}{2}} \exp\left(-\frac{\mathrm{i}fxM}{\hat{c}(\infty)}\right)\right]\left(1+\frac{u_1}{x}+\frac{u_2}{x^2}+\ldots\right), \end{aligned} \tag{3.23}$$

where the sign of the exponential has been chosen to ensure outgoing waves. Thus we are asserting that, because the velocity of the undisturbed vortex is contributing terms which are only $O(x^{-2})$, the expansion must start off like that for a uniform medium at rest.

Substitution of the expansions (3.23) into the governing equations (3.9) and expansion of A, B, C, D according to equation (3.21) leads to the equations

$$\left.\begin{aligned} Mp^* &= \rho(\infty)\,\hat{c}(\infty)\,u^*, \\ -\tfrac{1}{2}\mathrm{i}\frac{\hat{c}(\infty)}{fM}+p_1 &= u_1, \\ -\tfrac{3}{2}\mathrm{i}\frac{\hat{c}(\infty)}{fM}p_1+p_2 &= u_2+\frac{m}{f}-\frac{\beta}{\gamma-1}, \\ &\ldots\ldots \end{aligned}\right\} \tag{3.24}$$

arising from the first of (3.9) and

$$\left.\begin{aligned} Mp^* &= \hat{\rho}(\infty)\,\hat{c}(\infty)\,u^*, \\ -\tfrac{1}{2}\mathrm{i}\frac{\hat{c}(\infty)}{fM}+p_1 &= u_1, \\ \frac{m}{f}+\frac{\gamma\beta}{\gamma-1}-\frac{m^2\hat{c}(\infty)^2}{f^2M^2}+p_2 &= -\tfrac{1}{2}\mathrm{i}\frac{\hat{c}(\infty)}{fM}u_1+u_2, \\ &\ldots\ldots \end{aligned}\right\} \tag{3.25}$$

arising from the second. The consistency of the expansions (3.23) with the governing equations (3.9) is shown by the agreement of the first two equations in (3.24) and (3.25). If the third equation in (3.25) is subtracted from the third equation in (3.24) a second equation involving only p_1 and u_1 is obtained, so that p_1 and u_1 are determined.

If physical variables are restored, the resulting expression for P can be written in the form

$$P = p^* \left(\frac{a}{r}\right)^{\frac{1}{2}} \exp\left[-\frac{\mathrm{i}\omega r}{a_0(\infty)}\right] \left\{1 + \frac{\mathrm{i}a_0(\infty)}{\omega r}\left[-\frac{(4m^2-1)}{8} + \frac{a^2\omega^2}{a_0^2(\infty)}\left(\frac{m\Omega}{\omega} + \tfrac{1}{2}\beta\right)\right] + O\left(\frac{a_0^2(\infty)}{\omega^2 r^2}\right)\right\}, \quad (3.26)$$

where, in view of (3.22),
$$\beta = \frac{(\gamma-1)\,\Omega^2 a^2}{2a_0^2(a) + (\gamma-1)\,\Omega^2 a^2}. \quad (3.27)$$

If we let $\Omega \to 0$, keeping ω and $a_0(a)$ fixed, which corresponds to suppressing the vortex, equation (3.26) reduces to

$$P = \left[p^* \left(\frac{a}{r}\right)^{\frac{1}{2}} \exp\left(-\frac{\mathrm{i}\omega r}{a_0(\infty)}\right)\right] \left\{1 - \frac{ia_0(\infty)}{8\omega r}(4m^2-1) + O\left(\frac{a_0^2(\infty)}{\omega^2 r^2}\right)\right\}. \quad (3.28)$$

This expression times a constant agrees, to the indicated order, with the asymptotic expansion of

$$p^* H_m^{(2)}\left(\frac{\omega r}{a_0(\infty)}\right),$$

which is, provided that $\pi > \arg\omega > -2\pi$, the appropriate solution for an outgoing wave in a uniform gas at rest.

Evidently, the effect of the vortex flow field is contained in the second term in the square brackets in equation (3.26). The effect on the wave of the vortex flow field can be obtained from ray theory. To see this we note that in the absence of the vortex flow field the phase ψ of the wave satisfying the eikonal equation is constant at a moving point whose coordinates are $r_\psi = a_0(\infty)\,t$, $\theta_\psi = \theta_0$. Now if we combine the second term in the square brackets with the exponential then, with error $a_0^2(\infty)/r^2\omega^2$, the corresponding phase in the presence of flow is given by

$$\psi = \omega t + m\left(\theta + \frac{\Omega a^2}{r a_0(\infty)}\right) - \frac{\omega r}{a_0(\infty)}\left(1 - \frac{1}{2}\frac{\beta a^2}{r^2}\right). \quad (3.29)$$

Thus, since $r_\psi \sim a_0(\infty)\,t$, the phase is constant when

$$\left.\begin{aligned} \theta_\psi &\sim -\frac{\Omega a^2}{a_0^2(\infty)\,t}, \\ r_\psi &\sim a_0(\infty)\,t\left(1 + \frac{1}{2}\frac{\beta a^2}{a_0^2(\infty)\,t^2}\right). \end{aligned}\right\} \quad (3.30)$$

This result can be obtained directly from the equations of ray theory

$$\left.\begin{aligned} r_\psi \frac{\mathrm{d}\theta_\psi}{\mathrm{d}t} &= \frac{\Omega a^2}{r_\psi}, \\ \frac{\mathrm{d}r_\psi}{\mathrm{d}t} = a_0(r) &= a_0(\infty)\left\{1 - \frac{1}{2}\frac{\beta a^2}{r_\psi^2} + \dots\right\}, \end{aligned}\right\} \quad (3.31)$$

by integrating them approximately, noting $r_\psi \sim a_0(\infty)\,t$ to leading order. This provides a check on our analysis and shows that the vortex flow field affects the wave both directly by convection in the transverse direction and indirectly through its effect on the local sound speed.

Finally, we note that P and U must be continuous across the circle $x = 1$ where the vorticity changes discontinuously from its value 2 inside the vortex $(x < 1)$ to zero outside $(x > 1)$. This can be proved by integrating the governing equations (3.9) from $1-\epsilon$ to $1+\epsilon$ to find that since A, B, C and D are bounded functions

$$P(1+\epsilon)-P(1-\epsilon) = O(\epsilon),$$
$$U(1+\epsilon)-U(1-\epsilon) = O(\epsilon),$$

from which the result follows. Alternatively, we can derive the conditions from the requirements that the disturbed core boundary be a material surface across which the pressure $\bar{p}+\tilde{p}$ is continuous.

4. THE INCOMPRESSIBLE CASE $M = 0$

The eigenfrequencies for a constant vorticity core in an incompressible fluid were found by Kelvin (1880) and a summary of Kelvin's results can be found in Lamb (1931, p. 231) while the aero-acoustic significance of this solution has been examined by Broadbent (1976). These results guide all our subsequent work and so we review them here.

Kelvin found that there was just one mode of oscillation for each positive value of m and that the eigenfrequency was given by

$$f = 1-m \quad (m = 1, 2, 3, \ldots). \tag{4.1}$$

There is no solution for $m = 0$, because this would involve a non-zero mass flux across circles concentric with the vortex, and this is impossible in an incompressible fluid. The reason that there is just one mode, instead of a discrete spectrum of modes with differing radial structure, is that in the case of uniform vorticity, fixing the shape of the boundary fixes the vorticity distribution, and fixing the vorticity distribution fixes the velocity field. Since the vorticity in the core is uniform the disturbances are described everywhere by a velocity potential and it follows that

$$\left.\begin{aligned} U &= mx^{m-1} && (x < 1),\\ &= mx^{-m-1} && (x > 1).\end{aligned}\right\} \tag{4.2}$$

Then the θ component of the momentum equation (2.14) shows that

$$\left.\begin{aligned} P &= \mathrm{i}x^{m} && (x < 1),\\ &= \mathrm{i}(1-m+mx^{-2})\,x^{-m} && (x > 1).\end{aligned}\right\} \tag{4.3}$$

We remark that P vanishes when $\hat{\sigma}$ vanishes and that this occurs when

$$x = x_m = \left(\frac{m}{m-1}\right)^{\frac{1}{2}}. \tag{4.4}$$

Since in the work to follow f is close to its incompressible value (4.1), we can anticipate that P and $\hat{\sigma}$ will be small but finite for x close to the value $x = x_m$. We may remark that the phase relation between P and U changes as x increases through the value x_m, so that in this sense x_m is a critical layer. However, since the vorticity is zero, x_m is not a singular point of the perturbation equations, so that no special analytical treatment is called for.

Finally we note that $\tilde{v}_r$ and $\tilde{p}$ and $\tilde{v}_r$ and $\tilde{v}_\theta$ are exactly $\pi/2$ out of phase, so that the terms on the right hand side of the integral constraint (2.23) vanish as they must if the motion is to be purely oscillatory.

5. NUMERICAL DETERMINATION OF THE EIGENFREQUENCIES

We have chosen to work with a Riccati formulation of the governing equations (3.9). Thus we define

$$R = P/U, \tag{5.1}$$

and we find that

$$\mathrm{d}R/\mathrm{d}x = B+(A-D)\,R-CR^2. \tag{5.2}$$

In the incompressible case U never vanishes, except at 0 and ∞, and since we are seeking the modification to this oscillation caused by compressibility, we do not anticipate zero values of U in our solutions.

The singularity at $x = 0$ is avoided by use of a starting series derived from equations (3.20). This is

$$R(x,f) = r_0\,x+r_2\,x^3+O(x^5),$$

where the coefficients r_0 and r_2 can be shown to be

$$r_0 = -\mathrm{i}(f+m-2)\,\hat{\rho}(0)/m, \tag{5.3}$$

and

$$r_2 = -\frac{\mathrm{i}M^2\hat{\rho}(0)\,(f+m-2)}{2m^2(1+m)\,\hat{c}(0)^2}\,\{(f+m)^2-2f+m^2\}. \tag{5.4}$$

This formulation is quite satisfactory for $x < 1$ and *given* f the value $R(1,f)$ is easily found numerically.

If f differs from its incompressible value $1-m$ by a small complex number, $\hat{\sigma}$ will be small and hence A and C will be large for $x \sim x_m$; because $\hat{\omega} = 0$ for $x > 1$, B and D remain bounded. Now, as we pointed out in § 4, P will also be small for $x \sim x_m$, so that AP and CP will remain bounded. However, the fact that the computer is having to evaluate the ratio of two small numbers will cause loss of accuracy and it is better to eliminate the difficulty by a change of dependent variable.

We write

$$S = P/\hat{\sigma}U \tag{5.5}$$

and, putting

$$x = 1/y, \tag{5.6}$$

we find that

$$\mathrm{d}S/\mathrm{d}y = I+JS-KS^2, \tag{5.7}$$

where

$$I = \mathrm{i}\hat{\rho}/y^2, \tag{5.8}$$

$$J = -\frac{1}{y}-\frac{2M^2y}{\hat{c}^2}, \tag{5.9}$$

and

$$K = \frac{\mathrm{i}}{\hat{\rho}}\left(\frac{M^2\hat{\sigma}^2}{y^2\hat{c}^2}-m^2\right). \tag{5.10}$$

Equation (5.7) is singular at $y = 0$, but this can be avoided by invoking the asymptotic form of the solutions P and U for $x \gg 1$ to show that

$$S = \frac{\hat{c}(\infty)\,\hat{\rho}(\infty)}{Mf}\left\{1+\frac{\mathrm{i}\hat{c}(\infty)\,y}{2fM}+\left[\frac{\hat{c}(\infty)^2}{2f^2M^2}\,(m^2-\tfrac{3}{4})-\frac{\frac{1}{4}(\gamma+1)\,M^2}{1+\frac{1}{2}(\gamma-1)\,M^2}-\frac{m}{f}\right]y^2+\dots\right\}. \tag{5.11}$$

Thus for given f, we can find $S(1,f)$ by numerical integration. The boundary condition that P and U are continuous across $x = 1$ means that $\epsilon(f) = 0$, where

$$\epsilon(f) = R(1,f) - (f+m)\, S(1,f). \tag{5.12}$$

The function $\epsilon(f)$ is analytic, so that Newton's method can be used to find the roots. However a linear interpolation method which produces an improved estimate of f from two guesses works equally well.

Two independent computer programs were used: a double-precision real arithmetic program run on the ICL 1906S at R.A.E., Farnborough, and a single-precision complex arithmetic program run on the CDC 6500 at Imperial College. Fourth order Runge–Kutta integration was used, the R.A.E. program being designed to choose the integration step to meet predetermined error requirements. We compared the results obtained with the Riccati formulation with those found by integration of the original pair of linear equations and found that the second method gave much poorer accuracy.

One point about the numerical work is worth noting. The solution of the Riccati equation (5.7) which satisfies the boundary condition

$$S(0) = \frac{\hat{c}(\infty)\,\hat{\rho}(\infty)}{Mf}, \tag{5.13}$$

and which corresponds to an outgoing wave, is a smoothly varying function in $0 \leqslant y \leqslant 1$. However, this does not mean that its determination is free from difficulty. Suppose a small error is made in satisfying the boundary condition (5.13). If the corresponding solution is $S(y) + \eta(y)$ then

$$\frac{d\eta}{dy} = \eta\{J - 2KS\}, \tag{5.14}$$

and for $y \ll 1$, this differential equation becomes

$$\frac{d\eta}{dy} = -\frac{2iMf\eta}{\hat{c}(\infty)\,y^2} + O\left(\frac{1}{y}\right), \tag{5.15}$$

so that

$$\eta(y) = \alpha \exp\left(\frac{2iMf}{\hat{c}(\infty)\,y}\right), \tag{5.16}$$

where α is a constant. Thus when f is real the error is bounded, but oscillates rapidly; this means that the solution no longer represents a pure outgoing wave. The integration steps must be small enough to enable (5.15) to be integrated correctly or there is a risk† of the error growing exponentially. Thus if the integration is started at $y_s \ll 1$ using the starting series (5.11) then the integration step dy must satisfy

$$dy \ll \frac{c(\infty)\,y_s^2}{2Mf}. \tag{5.17}$$

The transformation (5.6) compresses the acoustic far field into the range $0 < y \ll M$ and we have to compensate for this in our choice of integration step.

The eigenvalues are shown in table 1. The main effect of compressibility is to make the vortex unstable. The growth rate, however, remains very small throughout the range of Mach numbers studied. The oscillation frequency is slightly reduced by compressibility below its incompressible

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† Numerical experiments reveal rapid blow up if (5.17) is violated.

value $|1-m|$. The cause of the instability appears to be the destruction of the $\pi/2$-out-of-phase relation between v_r and v_θ by the compressibility or, more precisely, by the capability of the system to radiate. This interpretation is supported by calculating the terms in the energy-like equation (2.23). The results are shown for two Mach numbers in figure 1 after non-dimensionalizing according to equation (3.7). A noticeable feature of the graph at $M = 1.0$ is that the acoustic radiation in the far field falls with increasing radius. This effect results from the fact that the unsteady motion at large radius had its origin in a disturbance of the vortex core at some earlier time. For stable motion, in which the amplitude decayed with time, the physical variables such as $\tilde{v}_r(r)$ would therefore be exponentially large in the far field, and conversely for the unstable motion actually found they are exponentially small. Moreover, the exponential decay rate is proportional to the magnitude of f_i, which table 1 shows to be 75 times larger at $M = 1$ than at $M = 0.2$, so that the effect is not apparent at the lower Mach number.

Table 1. $f = f_r + if_i$

		$m = 2$		
M	f_r	f_i	$(1+f_r)/M^2$	f_i/M^4
0.01	-0.99999167	-0.9806×10^{-9}	0.0833	-0.09806
0.02	-0.9999667	-0.1564×10^{-7}	0.08325	-0.09776
0.03	-0.9999251	-0.7883×10^{-7}	0.08319	-0.09732
0.04	-0.999867	-0.2477×10^{-6}	0.08311	-0.09675
0.05	-0.999792	-0.6004×10^{-6}	0.08302	-0.09606
0.1	-0.999175	-0.9129×10^{-5}	0.08252	-0.09129
0.2	-0.99672	-0.1245×10^{-3}	0.08192	-0.07781
0.5	-0.9787	-0.2378×10^{-2}	0.08504	-0.03804
1.0	-0.9089	-0.9236×10^{-2}	0.09109	-0.00924
1.5	-0.7994	-1.1559×10^{-2}	0.08917	-0.00228
2.0	-0.6676	-1.0074×10^{-2}	0.08310	-0.00063

	$m = 3$		$m = 4$	
M	f_r	f_i	f_r	f_i
0.1	-1.999168	-0.2287×10^{-6}	-2.999250	-0.4593×10^{-8}
0.2	-1.99669	-0.1097×10^{-4}	-2.99701	-0.7766×10^{-6}
0.6	-1.9703	-0.001542	-2.9732	-0.0005107
1.0	-1.9137	-0.006094	-2.9237	-0.003197
1.4	-1.8256	-0.009670	-2.8450	-0.006443
2.0	-1.6346	-0.009653	-2.6635	-0.007759

	$m = 6$		$m = 8$	
M	f_r	f_i	f_r	f_i
0.2	-4.99762	-0.3310×10^{-8}	-6.99806	-0.1317×10^{-10}
0.6	-4.9785	-0.4800×10^{-4}	-6.9824	-0.4223×10^{-5}
1.0	-4.9397	-0.0007411	-6.9507	-0.0001595
1.4	-4.8786	-0.002354	-6.9013	-0.0007821

For an incompressible vortex, it has already been noted that there is no solution for $m = 0$ and that for $m = 1$ is trivial, but these comments no longer apply when compressibility is allowed for. It is therefore possible that solutions exist for $m = 0$ and $m = 1$, but none was in fact found in the course of a few numerical trials.

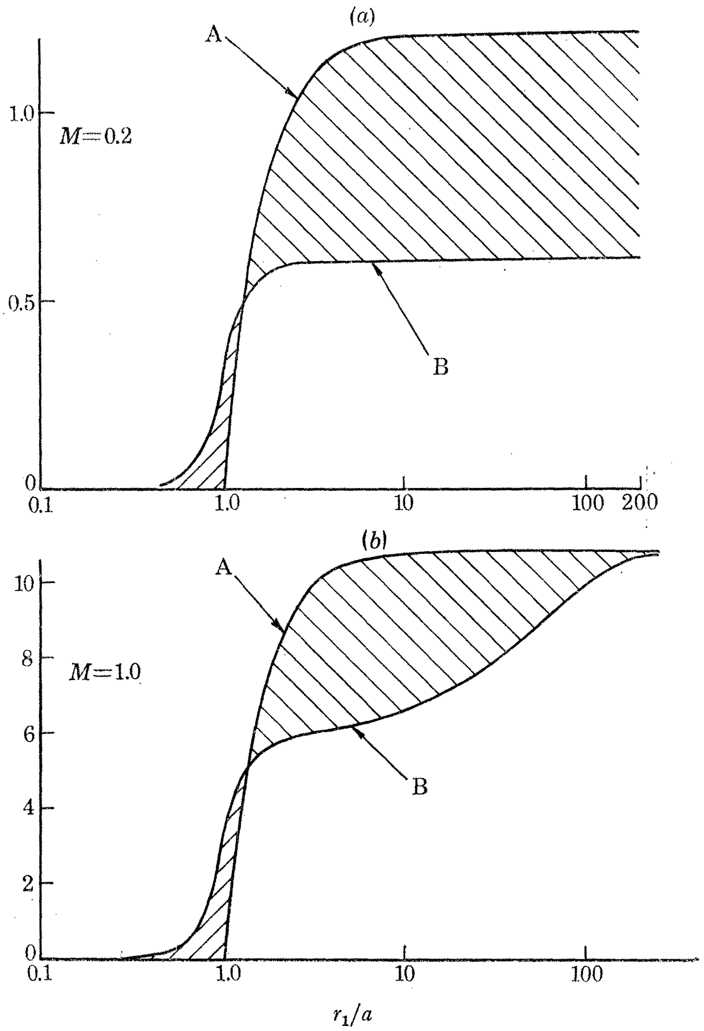

FIGURE 1. Balance of energy-like terms according to equation (2.24).

Curves A: $$-\frac{\int_0^{r_1}\bar{\rho}\langle\tilde{v}_r\tilde{v}_\theta\rangle\left(\frac{\mathrm{d}\bar{V}}{\mathrm{d}r}-\frac{\bar{V}}{r}\right)r\,\mathrm{d}r}{a^4\bar{\rho}(a)\,\Omega_0^3\exp\left(-2f_i\,\Omega_0\,t\right)}.$$

Curves B: $$\frac{\frac{\partial}{\partial t}\int_0^{r_1}\langle\tilde{E}\rangle r\,\mathrm{d}r}{a^4\bar{\rho}(a)\,\Omega_0^3\exp\left(-2f_i\,\Omega_0\,t\right)}.$$

The contribution of the radiation term

$$-[r\langle\tilde{v}_r\tilde{p}\rangle]_{r=r_1}/a^4\bar{\rho}(a)\,\Omega_0^3\exp\left(-2f_i\,\Omega_0\,t\right)$$

is indicated by cross-hatching: ▨ shows a positive contribution (energy inflow) and ▧ shows a negative contribution (energy outflow). In $x < 1$ the Reynolds stress

$$-\bar{\rho}\,\langle\tilde{v}_r\,\tilde{v}_\theta\rangle\left(\frac{\mathrm{d}\bar{V}}{\mathrm{d}r}-\frac{\bar{V}}{r}\right)\Big/\Omega_0$$

vanishes so that the energy inflow provides the whole increase in $\langle\tilde{E}\rangle$.

6. Analytical solution for small M

When the Mach number M is zero the velocity field and corresponding eigenfrequency are known exactly, so that we ought to be able to determine the velocity field and eigenfrequency for small M by a perturbation method. However the analysis is complicated by the fact that, even for $M \ll 1$, compressibility effects have to be allowed for in the 'far-field' region $Mx = O(1)$. Different expansions have to be introduced for $x = O(1)$ and $Mx = O(1)$ and matched in the intervening region. A thorough discussion of this method in the acoustic context has been given by Lesser & Crighton (1975).

We will consider only the case $m = 2$ and thus we can expand the eigenfrequency in the form

$$f = -1 + M^2 f_1 + M^4 f_2 + \dots; \tag{6.1}$$

the expansion is essentially in powers of M^2, as is evident from the dependence of A, B, C and D on M. However we do *not* assert that f_s is $O(1)$ as $M \rightarrow 0$ and in fact f_2 proves to involve $\log M$. All that we really require is $M^2 f_s \ll f_{s-1}$ as $M \rightarrow 0$. We could adopt a more formal approach using gauge functions (see Lesser & Crighton) but this is not necessary here. The Riccati formulation again proves helpful, provided that the difficulty of the near-simultaneous vanishing of P and $\hat{\sigma}$ is avoided by the device explained in § 5.

It is natural to divide the near field into two regions; $0 \leqslant x \leqslant 1$ corresponding to the vortex core and $1 \leqslant x \ll M^{-1}$.

For $0 \leqslant x \leqslant 1$ we expand R in the form

$$R = R_0 + M^2 R_1 + M^4 R_2 + \dots, \tag{6.2}$$

where

$$R_0 = \tfrac{1}{2}\mathrm{i}x \tag{6.3}$$

is the incompressible solution given in equations (4.2) and (4.3). This solution has to satisfy the condition

$$R(x) = r_0 x + O(x^3) \quad \text{as} \quad x \rightarrow 0, \tag{6.4}$$

where r_0 is defined in equation (5.3), so that after some algebra, we find that, as $x \rightarrow 0$,

$$\left.\begin{aligned} R_0 &= \tfrac{1}{2}\mathrm{i}x + O(x^3), \\ R_1 &= -\tfrac{1}{2}\mathrm{i}(\tfrac{1}{2} + f_1)\, x + O(x^3), \\ \text{and} \qquad R_2 &= \tfrac{1}{2}\mathrm{i}(\tfrac{1}{4} - \tfrac{1}{8}\gamma - f_2 + \tfrac{1}{2} f_1)\, x + O(x^3). \end{aligned}\right\} \tag{6.5}$$

If the coefficients of the differential equation (5.2) satisfied by R are expanded to the requisite order in M^2 and the expansion (6.2) substituted, then equations for R_0, R_1, R_2, ... can be derived. R_0 satisfies the equation

$$\frac{\mathrm{d}R_0}{\mathrm{d}x} = 3\mathrm{i} - \frac{7R_0}{x} - \frac{4\mathrm{i}R_0^2}{x^2}, \tag{6.6}$$

and it is easy to verify that the incompressible solution (6.3) satisfies this equation and the boundary condition (6.5).

Since the full equation is nonlinear, the (linear) differential equation for R_s involves R_{s-1}, R_{s-2}, ... R_0. Thus now that R_0 is determined the differential equation for R_1 can be found, namely

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^3 R_1) = \tfrac{7}{4}\mathrm{i}x^5 - \mathrm{i}x^3(1 + 2f_1). \tag{6.7}$$

If this is integrated, then

$$R_1 = \frac{7\mathrm{i}}{24}x^3 - \frac{\mathrm{i}x}{4}(1+2f_1) + \frac{C}{x^3}, \tag{6.8}$$

where C is a constant of integration. The boundary condition (6.5) shows that $C = 0$, so that R_1 is determined. The equation for R_2 can now be determined and after some lengthy algebra

$$R_2 = \frac{\mathrm{i}}{8}\left(\frac{187}{144} - \frac{5\gamma}{8}\right)x^5 + \frac{7\mathrm{i}}{48}(\gamma - 2f_1 - 2)\,x^3 + \frac{\mathrm{i}}{4}\left(\frac{1}{2} - \frac{\gamma}{4} + f_1 - 2f_2\right)x. \tag{6.9}$$

The region $M \ll y < 1$ is dealt with by expanding S in the form

$$S = S_0 + M^2 S_1 + M^4 S_2 + \dots, \tag{6.10}$$

where

$$S_0 = \frac{\mathrm{i}}{2y} \tag{6.11}$$

is the incompressible solution. If (6.10) and (6.11) are substituted into the governing equation (5.7) and, keeping y fixed and $O(1)$, the functions I, J and K expanded in powers of M^2 we find that S_1 satisfies the equation

$$\mathrm{d}(y^5 S_1)/\mathrm{d}y = -\mathrm{i}y^5 + \tfrac{1}{4}\mathrm{i}y \tag{6.12}$$

so that

$$S_1 = -\frac{\mathrm{i}y}{6} + \frac{\mathrm{i}y^{-3}}{8} + Cy^{-5} \tag{6.13}$$

where C is a constant of integration.

The constant C must be determined by matching to the far-field solution. Fortunately, we do not need the details of the far-field solution to see that C must be zero. If y is $O(M)$ then S_1 is $O(CM^{-5})$ so that S is $O(CM^{-3})$. However, it is clear from the expansion (5.11) that S is $O(M^{-1})$ in the far field; thus C must be zero.

Now that S_1 is determined the differential equation satisfied by S_2 can be found and, after a lengthy calculation, we find that

$$S_2 = \mathrm{i}\left[\frac{y^{-5}\ln y}{16} + \left(-\frac{1}{8} - \frac{f_1}{4} - \frac{\gamma}{16}\right)y^{-3} + \left(\frac{1}{16} + \frac{3\gamma}{32} + \tfrac{1}{4}f_1\right)y^{-1} - \tfrac{1}{9}y + \left(\frac{23}{288} - \frac{\gamma}{32}\right)y^3\right] + \mathrm{i}By^{-5}, \tag{6.14}$$

where $\mathrm{i}B$ is a constant of integration. The term involving the constant of integration makes a contribution to S of $O(BM^{-1})$ when y is $O(M)$. Thus we can find B only by a detailed match to the far-field solution, which we must now discuss.

We have already noted that in the far field y is $O(M)$ and S is $O(M^{-1})$. Thus variables appropriate to the far field are

$$\left.\begin{aligned} \bar{S} &= MS, \\ \text{and} \qquad \bar{y} &= M^{-1}y. \end{aligned}\right\} \tag{6.15}$$

If we substitute into the Riccati equation (5.7) we find that

$$\mathrm{d}\bar{S}/\mathrm{d}\bar{y} = \bar{I} + \bar{J}\bar{S} - \bar{K}\bar{S}^2, \tag{6.16}$$

where $\bar{I}$, $\bar{J}$ and $\bar{K}$ can easily be determined. For fixed $\bar{y}$ we can show that

$$\bar{I} = \frac{\mathrm{i}}{\bar{y}^2}(1 + \tfrac{1}{2}M^2 + O(M^4)), \tag{6.17}$$

$$\bar{J} = -\frac{1}{\bar{y}} + O(M^4), \tag{6.18}$$

and

$$\bar{K} = \mathrm{i}\left[\frac{1}{\bar{y}^2} - 4 - \frac{M^2}{\bar{y}^2}\left(2f_1 + \tfrac{1}{2}\gamma + 2\bar{y}^2\right)\right] + O(M^4). \tag{6.19}$$

Thus, if we expand $\bar{S}$ in the form

$$\bar{S} = \bar{S}_0 + M^2\bar{S}_1 + M^4\bar{S}_2 + \ldots, \tag{6.20}$$

we find that

$$\frac{\mathrm{d}\bar{S}_0}{\mathrm{d}\bar{y}} = \frac{\mathrm{i}}{\bar{y}^2} - \frac{1}{\bar{y}}\bar{S}_0 - \mathrm{i}\left(\frac{1}{\bar{y}^2} - 4\right)\bar{S}_0^2, \tag{6.21}$$

while from the radiation condition embodied in equation (5.11)

$$S(0) = \frac{\hat{c}(\infty)\,\hat{\rho}(\infty)}{Mf}. \tag{6.22}$$

Thus

$$\bar{S}_0 \sim -1 \quad \text{as} \quad \bar{y} \to 0, \tag{6.23}$$

since $f = -1 + O(M^2)$, $\hat{c}(\infty) = 1 + O(M^2)$ and $\hat{\rho}(\infty) = 1 + O(M^2)$. A scrutiny of the derivation of (6.20) reveals that the effects of rotation do not occur at this order, which suggests that S_0 should correspond to an outgoing sound wave in a uniform medium. This would have the pressure behaving like

$$H_2^{(2)}\left(\frac{\omega r}{a_0(\infty)}\right),$$

provided that $\pi > \arg\omega > -2\pi$. Moreover $\omega \sim \mathrm{e}^{-\pi\mathrm{i}}\Omega_0[1 + O(M^2)]$ so that, in terms of outer variables,

$$P = P_0 H_2^{(2)}\left(\frac{\mathrm{e}^{-\pi\mathrm{i}}}{\bar{y}}\right) = -P_0 H_2^{(1)}\left(\frac{1}{\bar{y}}\right), \tag{6.24}$$

where P_0 is a constant. The radial equation of motion shows that

$$U = \mathrm{i}P_0 M H_2^{(1)\prime}\left(\frac{1}{\bar{y}}\right), \tag{6.25}$$

leading to

$$\bar{S}_0 = -\frac{\mathrm{i}H_2^{(1)}(1/\bar{y})}{H_2^{(1)\prime}(1/\bar{y})}. \tag{6.26}$$

Having constructed this solution we can easily verify that it satisfies the differential equation (6.21) and the boundary condition (6.23).

If we expand the solution (6.26) in ascending powers of $1/\bar{y}$ we find that

$$\bar{S}_0 = \frac{\mathrm{i}}{2}\bar{y}^{-1} + \frac{\mathrm{i}}{8}\bar{y}^{-3} + \frac{\mathrm{i}}{16}\bar{y}^{-5}\ln\bar{y} + \mathrm{i}\tau\bar{y}^{-5} + O(\bar{y}^{-7}\ln\bar{y}), \tag{6.27}$$

where

$$\tau = \frac{\ln 2}{16} - \frac{c}{16} + \frac{1}{32} + \frac{\pi\mathrm{i}}{32}; \tag{6.28}$$

c is Euler's constant. Now if we put the inner expansion in outer variables we find that

$$M\{S_0 + M^2S_1 + M^4S_2\} = \frac{\mathrm{i}}{2}\bar{y}^{-1} + \frac{\mathrm{i}}{8}\bar{y}^{-3} + \frac{\mathrm{i}}{16}\bar{y}^{-5}\ln\bar{y} + \mathrm{i}\bar{y}^{-5}\left(\tfrac{1}{16}\ln M + B\right) + O(M^2). \tag{6.29}$$

If the expansions (6.27) and (6.29) are to agree, then

$$B + \tfrac{1}{16}\ln M = \tau. \tag{6.30}$$

The agreement of the remaining terms shows that the matching procedure is working, although we have no guarantee that it will not fail at higher order. One point should be made. The higher terms in the inner expansion M^7S_3, M^9S_4, ... will make an $O(1)$ contribution when expressed in outer variables. However this contribution will come from the y^{-7}, y^{-9}, ... terms in the solutions for S_3, S_5, ... and must therefore match higher terms in the expansion (6.27). The determination of B will not be affected.

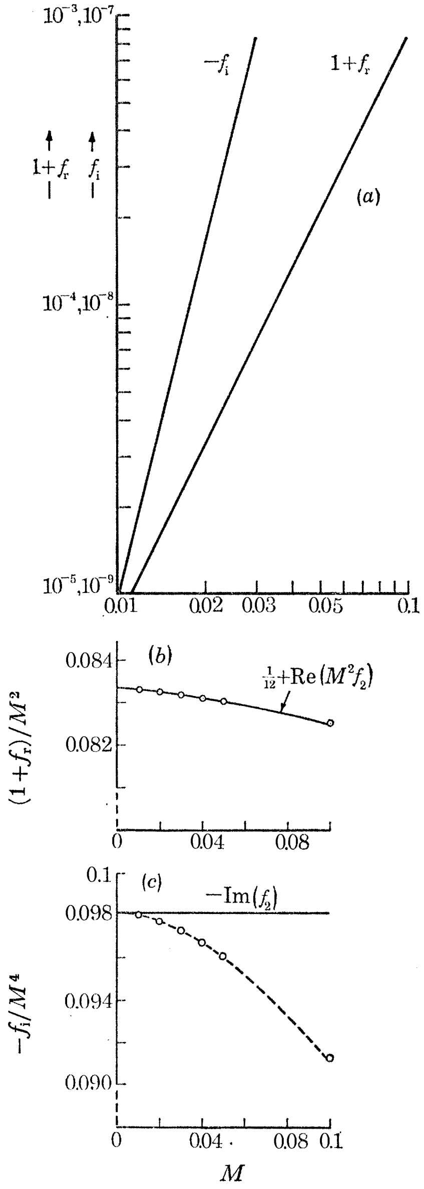

FIGURE 2. (*a*) Logarithmic plot of the real and imaginary parts of f ($f \equiv f_r + \mathrm{i}f_i$) against M for $m = 2$; it may be verified that the gradients of $1+f_r$ and $-f_i$ are respectively 2 and 4 in agreement with the lowest-order terms in the expansion (6.1) according to (6.33) and (6.34). (*b*) Variation of $(1+f_r)/M^2$ with M. The full line is taken from equations (6.33) and (6.34) and the circles are results from the numerical solutions (table 1). (*c*) Variation of $-f_i/M^4$ with M. The full line is taken from equation (6.34) and the circles are numerical results. The analysis of § 6 indicates that $\mathrm{Im}\,f_3$ would contain a constant and a log M term, so a curve $f_i/M^4 = a + bM^2 + cM^2 \ln M$ has been fitted through the circles at $M = 0.01$, 0.02, 0.03, and is shown dashed. The fitted values are $a = -0.0981752$, $b = 0.237098$, $c = -0.203522$; the value for a may be compared with that found analytically in (6.34) which is -0.0981748.

The solutions R_0, R_1, R_2 and S_0, S_1, S_2 are now completely determined in terms of the unknown frequency shifts f_1 and f_2 introduced in equation (6.1). Expansion of the boundary condition

$$R(1,f) = (f+2)\,S(1,f) \tag{6.31}$$

in powers of M leads to

$$\left.\begin{aligned} R_0(1) &= S_0(1), \\ R_1(1) &= S_1(1)+f_1 S_0(1), \\ \text{and}\qquad R_2(1) &= S_2(1)+f_1 S_1(1)+f_2 S_0(1). \end{aligned}\right\} \tag{6.32}$$

Thus f_1 and f_2 are determined and

$$f_1 = \tfrac{1}{12} \tag{6.33}$$

and

$$f_2 = -\tfrac{1}{16}\ln\left(\frac{2}{M}\right)+\frac{\gamma}{192}+\frac{67}{1162}+\frac{c}{16}-\frac{\pi \mathrm{i}}{32}. \tag{6.34}$$

A comparison between f calculated in this way and the values obtained numerically is shown in figure 2, for the case $\gamma = 1.4$. The agreement between the real parts is very close over the range shown, whereas the imaginary parts agree closely only when M is very small, suggesting that the M^6 term has relatively large coefficients; values obtained by curve fitting are given in the caption to figure 2(*c*).

If we work back through the calculation we can see that f_2 has an imaginary part because the constant τ has an imaginary part. This in turn is a consequence of the fact that we need a Hankel function of the first kind to comply with the radiation condition and the ascending series expansion of this function has complex coefficients. There is thus an intimate connection between the occurrence of the instability and the ability of the system to radiate; the $\pi/2$-out-of-phase relation between P and U characteristic of the incompressible solution is destroyed and pure oscillations are impossible.

7. Further comments

The main numerical results are plotted in figure 3 and show that $m = 2$ is always the most unstable mode, although at high Mach numbers $m = 3$ is nearly as unstable. It is natural to inquire how these results are affected by the nature of the velocity profile of the vortex and by the possibility of three-dimensional disturbances of the type $\exp\{\mathrm{i}(\omega t+m\theta+kz)\}$. It is hoped to examine these effects more fully in a later paper, but preliminary numerical results indicate that a simple rounding of the velocity profile near $x = 1$ does not remove the instability (in fact for the same total circulation the growth rate was slightly increased) and that modes with $k > 0$ are less unstable than those with $k = 0$.

The mode of oscillation we have studied is a compressible modification of the unique incompressible oscillation. It is possible that compressibility introduces further modes with more complicated radial structure. However, we have found no evidence of these, although we must stress that we have not undertaken the considerable numerical work involved in a systematic search for them.

The significance of the instability in the noise radiated from a shear layer such as that of a jet is a matter for debate. There is by now considerable evidence that large vortex-like structures do exist in a shear layer and that if they come under sufficient strain they become unstable and may disintegrate (Moore & Saffman 1971, 1975). Such events would undoubtedly be powerful

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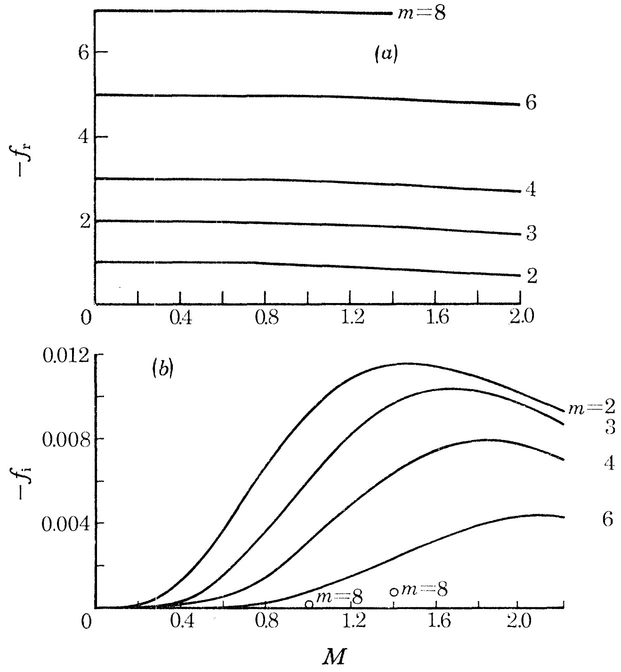

FIGURE 3. Variation of the real and imaginary parts of f with M, for various values of m.

radiation centres of noise. The present results suggest that the vortices may be more unstable than had previously been thought, since compressibility allows perturbations to extract energy from the mean flow by virtue of the capacity to radiate, and this would no doubt influence the acoustic radiation of a vortex in a strain field. On the other hand the magnitudes of the instabilities found are too small to have much direct significance in the behaviour of a turbulent shear layer.

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